Indian J. Phys. **43**, 762–766 (1969)

On Kelvin-Helmholtz instability in the presence of a uniform electric field

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(Received 6 June 1969).

The interfacial instability arising due to relative motion of two fluids in the presence of a uniform electric field is studied. The parameters involved are a, b, c which denote the relative velocities, the electric part of the Maxwellian stress and the surface tension, respectively. It is shown that in the absence of surface tension and if $a+b>1$ the system is unstable for all wave numbers. Moreover, if $a+b<1$ the system is stable for wave numbers less than the critical wave number x_0 beyond which the system is unstable. Further, we note that in the presence of surface tension when $a+b>1$ the system is unstable below the critical wave number x'_0 above which the system is stable. It has been found that the increase in surface tension decreases the wave number band. On the other hand, if $a+b<1$ the system is always stable for all permissible C.

Introduction

The stability of a horizontal fluid interface between a conducting and a non-conducting fluid in the presence of transverse dc electric field has been studied experimentally by Taylor & McEwan (1965). It has been shown that the interface becomes unstable under the action of a sufficiently great electric field. Yih (1968) extended the investigation to ac electric fields and showed that the interface can be unstable even if the electric field is at all times weaker than that needed for stability in the case of a steady fieid, and that when instability occurs the waves may either be synchronous with the electric field or have twice its frequency. The purpose of this note is to study the stability of the interface when the fluids are in relative horizontal motion. The analysis takes into account the hydrodynamics and electrodynamics through the continuum picture.

We consider incompressible inviscid fluids. The upper fluid with constant density ρ_1 is taken to be nonconducting and the lower with density ρ_2 a perfect conductor. The XOY plane is taken to coincide with the unperturbed middle level of the layer and the positive Z-axis in the upward direction normal to the unperturbed fluid surfaces. The upper fluid is bounded above by an electrode at $z=h_1$ with potential ϕ_{10} and below by the interface, and the lower fluid is bounded below by an electrode at $z=-h_2$ with potential ϕ_{20}. Further, we assume that initially the non-conducting and conducting fluids are moving with velocities $\vec{V}_1$ and $\vec{V}_2$, respectively, in the X-direction.

BASIC EQUATIONS AND EQUILIBRIUM STATE

The basic equations of the problem for the non-conducting and conducting fluids are as follows

Non-conducting :

$$\operatorname{div} \vec{v}_1=0, \quad \ldots(2.1)$$

$$\rho_1\left[\frac{\delta \vec{v}_1}{\delta t}+(\vec{v}_1 \nabla)\vec{v}_1\right]=-\operatorname{grad} P_1+\frac{K}{8\pi}\operatorname{grad} \vec{E}_1^{\,2}-\rho_1 g, \quad \ldots(2.2)$$

$$\nabla^2\phi_1=0 \quad \ldots(2.3)$$

$$\vec{g}=(0,0,g)$$

where $\vec{E}_1=-\operatorname{grad}\phi_1$, is the electric field and K is the dielectric constant.

Conducting :

$$\operatorname{div} \vec{v}_2=0, \quad \ldots(2.4)$$

$$\rho_2\left[\frac{\delta \vec{v}_2}{\delta t}+(\vec{v}_2 \nabla)\vec{v}_2\right]=-\operatorname{grad} P_2-\rho_2\vec{g}, \quad \ldots(2.5)$$

$$\vec{g}=(0,0,g).$$

In the equilibrium state

$$\vec{v}_1=(v_1,0,0) \text{ and } \vec{v}_2=(v_2,0,0) \quad \ldots(2.6)$$

The pressure distributions in the two fluids are given as

$$P_{10}=P_0-\rho_1 gz, \quad \ldots(2.7)$$

$$P_{20}=P_0-\frac{KE_0^{\,2}}{8\pi}-\rho_2 gz, \quad \ldots(2.8)$$

where, P_0 is the hydrostatic pressure at the interface $z=0$.

The electric potential ϕ_0 is ϕ_{20} in the conducting fluid so that the electric field in the fluid is zero. In the non-conducting fluid

$$\phi_0=\phi_{20}+\frac{\phi_{10}-\phi_{20}}{h_1}z \quad \ldots(2.9)$$

so the vertical electric field in that fluid is

$$\vec{E}_0=\frac{\phi_{20}-\phi_{10}}{h_1} \quad \ldots(2.10)$$

LINEARIZED EQUATIONS AND BOUNDARY CONDITIONS

Assuming that the perturbed quantities depend on time t and spatial co-ordinates x and y on $\exp\iota(\omega t+kx+ly)$, the linearized equations for the non-conducting and conducting fluids are as follows

$$\nabla \cdot \vec{v}_{11}=0, \quad \text{...(3.1)}$$

$$\rho_{11} \cdot \iota(\omega+kV_1)\,\vec{v}_{11}=-\nabla\left(P_{11}-\frac{K}{4\pi}\vec{E}_1\cdot\vec{E}_0\right), \quad \text{...(3.2)}$$

$$\nabla^2\phi_{11}=0, \quad \text{...(3.3)}$$

$$\nabla \cdot \vec{v}_{21}=0, \quad \text{...(3.4)}$$

$$\rho_{21} \cdot \iota(\omega+kV_2)\,\vec{v}_{21}=-\nabla P_{21} \quad \text{...(3.5)}$$

Let $z=\delta z e^{\iota(\omega t+kx+ly)}$ be the equation of the perturbed surface. The boundary conditions satisfied by the electric potential are

$$\phi_1=\phi_0 \quad \text{at } z=h_1$$

$$\phi_1=\phi_2 \quad \text{at } z=\delta z, \text{ perturbed surface}$$

i.e.

$$\left.\begin{aligned} &\phi_{11}=0 \quad \text{at } z=h_1 \\ &\phi_{11}=-\delta z\left(\frac{\phi_{10}-\phi_{20}}{h_1}\right) \text{ at } z=0, \text{ the unperturbed surface}\end{aligned}\right\} \quad \text{...(3.6)}$$

and

The kinematic boundary conditions are

$$\frac{v_{z_{11}}}{\iota(\omega+kV_1)}=\frac{v_{z_{21}}}{\iota(\omega+kV_2)} \text{ at } z=0, \quad \text{...(3.7)}$$

$$v_{z_{11}}=0 \quad \text{at } z=h_1 \quad \text{...(3.8)}$$

$$v_{z_{21}}=0 \quad \text{at } z=-h_2 \quad \text{...(3.9)}$$

The dynamic boundary condition is

$$P_{11}-P_{21}+\delta z(\rho_2-\rho_1)g-\frac{K}{4\pi}\vec{E}_1\cdot E_0=T\nabla^2\delta z, \text{ at } z=0, \quad \text{...(3.10)}$$

T being the surface tension.

Dispersion relation

From equation (3.1) — (3.5) on solving we get

$$v_{z_{11},\,21}=\alpha(C_{11,\,21}\,e^{\alpha z}-C_{21,\,22}\,e^{-\alpha z}), \quad \text{...(4.1)}$$

$$P_{11,\,21}=-\iota\,\rho_{1,\,2}\,(\omega+kV_{1,2})\,(C_{11,21}\,e^{\alpha z}+C_{21,\,22}\,e^{-\alpha z}), \quad \text{...(4.2)}$$

$$\phi_{11}=(A_1e^{\alpha z}+B_1e^{-\alpha z}), \quad \text{...(4.3)}$$

where $\alpha=\sqrt{k^2+l^2}$, C_{11}, C_{21}, C_{22}, A_1, B_1 are arbitrary constants.

The six boundary conditions (3.6) — (3.10) give six homogeneous equations for the six arbitrary constants. The eliminant of these gives the following dispersion relation

$$(\omega+kV_1)^2\rho_1\coth\alpha h_1+(\omega+kV_2)^2\rho_2\coth\alpha h_2$$
$$+\alpha\left\{\frac{K\alpha}{4\pi}\left(\frac{\phi_{10}-\phi_{20}}{h_1}\right)^2\coth\alpha h_1-(\rho_2-\rho_1)g-\alpha^2T\right\}=0 \quad \text{...(4.4)}$$

which gives

$$\omega(\rho_1 \coth \alpha h_1+\rho_2 \coth \alpha h_2)+k(\rho_1 V_1 \coth \alpha h_1+\rho_2 V_2 \coth \alpha h_2) = \pm\iota\,[X]^{\frac{1}{2}} \qquad ..\ (4.5)$$

where
$$X=\frac{k^2\rho_1 \,.\, \rho_2 \coth \alpha h_1 \coth \alpha h_2\,(V_1-V_2)^2}{\alpha(\rho_1 \coth \alpha h_1+\rho_2 \coth \alpha h_2)}$$
$$+\frac{K\alpha}{4\pi}\left(\frac{\phi_{10}-\phi_{20}}{h_1}\right)^2 \coth \alpha h_1-(\rho_2-\rho_1)\,g-\alpha^2\,T. \qquad ...(4.6)$$

Considering the wave motion in X-direction only, *i.e.* putting $l=0$ and $h_1=h_2$, X can be put in the following form

$$X=(a+b)\,x \coth x-(1+cx^2), \qquad ...(4.7)$$

where $kh_1=x$

$$\left.\begin{aligned} a&=\frac{\rho_1\,(V_1-V_2)^2}{h_1\left(1+\frac{\rho_1}{\rho_2}\right)(\rho_2-\rho_1)\,g},\\ b&=\frac{K}{4\pi\, h_1\,(\rho_2-\rho_1)\,g}\,.\,\left(\frac{\phi_{10}-\phi_{20}}{h_1}\right)^2,\\ c&=\frac{T}{h_1^2\,(\rho_2-\rho_1)\,g}\end{aligned}\right\} \qquad ..\ (4.8)$$

As $\rho_2 > \rho_1$, we note that from (4.7) and (4.5) the interface will be unstable if the function

$f(x)=(a+b)\,x \coth x-(1+cx^2)$, is positive and stable when it is negative. .. (4.9)

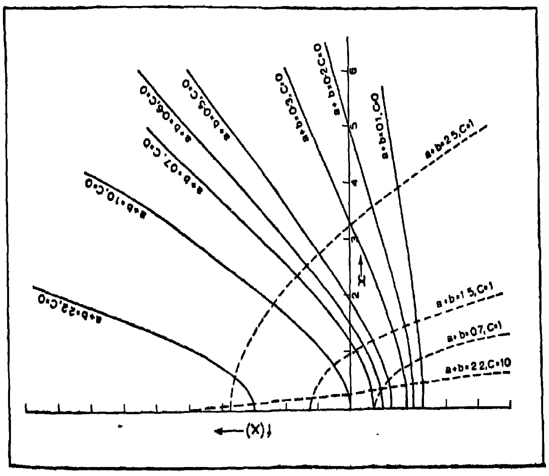

Figure 1.

The function $f(x)$ is plotted in figure 1 for various values of a, b and c. From the analytic behaviour of $f(x)$ and that shown in the figure we make the important conclusions in the following section.

Conclusions

Case I $c=0$.

$$f(x)=(a+b)x \coth x-1$$

From the numerical work done we conclude that if $(a+b)>1$ the system is unstable for all wave numbers. If $(a+b)<1$ the system is stable for wave numbers less than the critical wave number x_0, beyond which the system is unstable. We know the relative streaming between two fluids destabilizes short wave-length perturbations (3). The above results show that this is the case even when the electric part of the Maxwellian stress is present and further that this short wavelength instability is enhanced by the presence of Maxwellian stresses.

Case II Effect of surface tension

In this case we have

$$f(x)=(a+b)\, x \coth x-(1+cx^2),$$

For $(a+b)>1$ and for small values of c the system is unstable for $x<x'_0$ and then it is stable for $x>x'_0$. As we decrease the surface tension parameter c it appears taht the unstable wave number band is increased. For large c the wave number band is decreased.

When $(a+b)<1$ and whatever may be the value of c, the surface tension parmeter, the system is always stable. Thus we conclude that for $(a+b)>1$, and for any value of c the system has a stabilizing effect due to the presence of surface tension. In this case we may say that the surface tension inhibits the Kelvin-Helmholtz instability.

Acknowledgement

The author is grateful to Dr. C. Uberoi for suggesting the prōblem and her constant guidance during the preparation of this note. He is also grateful to Prof. P. L. Bhatnagar for his keen interest and kind encouragement.

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